DAMTP/96-38
SphT/96-033

A Renormalisation Group Analysis of 2d Freely Decaying Magnetohydrodynamic Turbulence

by

Ph. Brax*

DAMTP
University of Cambridge
Silver Street
Cambridge CB39EW UK

Abstract:

We study two dimensional freely decaying magnetohydrodynamic turbulence. We investigate the time evolution of the probability law of the gauge field and the stream function. Assuming that this probability law is initially defined by a statistical field theory in the basin of attraction of a renormalisation group fixed point, we show that its time evolution is generated by renormalisation transformations. In the long time regime, the probability law is described by non-unitary conformal field theories. In that case, we prove that the kinetic and magnetic energy spectra are proportional. We then construct a family of fixed points using the $(p, p+2)$ non-unitary minimal models of conformal field theories.

Pacs numbers: 47.27.Ak, 11.10.Gh, 05.40.+j

* On leave of absence from SPhT-Saclay CEA F-91191 Gif sur Yvette

It has been recently noticed[1] by Polyakov that field theoretic methods can be used to study steady states of two dimensional turbulence. In this approach, Polyakov assumes that the velocity field of a turbulent fluid can be described by a primary field deduced from a conformal field theory. The same method has been applied[2] by Ferretti and Yang in the case of steady states of magnetohydrodynamic turbulence. Other possible applications to magnetohydrodynamic have been also discussed by Coceal and Thomas[3]. One of the results found by Ferretti and Yang is that some steady states do not respect the equipartition [4] between the magnetic and the kinetic energies. It has been argued[5] by Rahimi-Tabar and Rouhani that the equipartition can be satisfied provided that time as well as spatial scale invariances hold. In this communication, we shall use renormalisation group techniques to study the long time behaviour of freely decaying magnetohydrodynamic turbulence in two dimensions. In particular, we shall see that conformal field theories appear as fixed points of the time evolution of the probability law of the stream function and the gauge field in the long time regime. At these fixed points, the kinetic and the magnetic energy spectra are proportional. Finally, we shall construct a family of fixed points using the $(p, p+2)$ non-unitary minimal models.

In two dimensions, the velocity field of an incompressible fluid is determined by a single pseudo-scalar field: the stream function

$$v_\alpha = e_{\alpha\beta}\partial_\beta\psi. \tag{1}$$

The tensor $e_{\alpha\beta}$ is the antisymmetric tensor $e_{12} = -e_{21} = 1$. The vorticity is then given by

$$\omega = e_{\alpha\beta}\partial_\alpha v_\beta. \tag{2}$$

It is a pseudo-scalar too. We are interested in the two dimensional situation where the magnetic field lies in the plane of the fluid motion. Moreover we suppose that there is no external magnetic field and the magnitude of the initial magnetic field is of the same order as the magnitude of the initial velocity field (in well-chosen units). This situation can occur for certain astrophysical phenomena where the magnetic Reynolds number is large. In the absence of magnetic monopoles, the magnetic field can also be represented by

$$B_\alpha = e_{\alpha\beta}\partial_\beta A, \tag{3}$$

where A is a scalar gauge field. The induced electric current can be deduced

$$J = e_{\alpha\beta}\partial_\alpha B_\beta. \tag{4}$$

These quantities satisfy coupled partial differential equations. We will focus on the free decaying case in an infinite plane. We shall suppose that the vorticity field and the electric current decay sufficiently fast at infinity.

The Navier-Stokes equations are effective equations describing the behaviour of a fluid in the hydrodynamic approximation. The velocity field is the average of the microscopic velocity of molecules over a macroscopic size a, the size of a fluid element. Similarly, the

magnetic field is also a macroscopic field. We shall analyse the macroscopic equations for the regularised vorticity and gauge fields[2]

$$\begin{aligned} \partial_t \omega_a + e_{\alpha\beta}\partial_\alpha \psi_a \partial_\beta \Delta \psi_a &= e_{\alpha\beta}\partial_\alpha J_a \partial_\beta A_a + \nu \Delta \omega_a \\ \partial_t A_a + v_{a\alpha}\partial_\alpha A_a &= \eta J_a. \end{aligned} \tag{5}$$

The viscosity ν and the resistivity η are responsible for dissipation. As the resistivity and the viscosity have the same dimensions (in well-chosen units) , one deduces

$$\begin{aligned} \nu &= \frac{a^2}{\tau} \\ \eta &= \eta_0 \frac{a^2}{\tau}, \end{aligned} \tag{6}$$

where η_0 is of order one. The time τ is a characteristic time for the decay of both the kinetic and the magnetic energies.

In field theoretic terms, the size a plays the role of an intrinsic ultraviolet cut-off. We shall be interested in the situation where the gauge field and the stream function are random fields regularised at short distances by the molecular size a. As no random stirring is imposed, we shall suppose that initial conditions are randomly prescribed by a probability law for both the initial magnetic and velocity fields. The evolution of a given configuration is given by (5). The evolution of the probability law of the gauge field and the stream function $d\mathcal{P}(A_a, \psi_a)$ is governed by the Hopf equations obtained considering (5) as valid in any equal time correlation functions of the gauge field and the stream function

$$\begin{aligned} &\frac{\partial}{\partial t} < \omega_a(x_1,t) ... \omega_a(x_n,t) A_a(x_{n+1},t) ... A_a(x_{n+m},t) > \\ &\sum_{i=1}^{n} < \omega_a(x_1,t) ... (-e_{\alpha\beta}\partial_\alpha \psi_a(x_i,t)\partial_\beta \Delta\psi_a(x_i,t) + e_{\alpha\beta}\partial_\alpha J_a(x_i,t)\partial_\beta A_a(x_i,t) \\ &+ \nu \Delta \omega_a(x_i,t)) ... \omega_a(x_n,t) A_a(x_{n+1},t) ... A_a(x_{n+m},t) > + \\ &\sum_{i=n+1}^{n+m} < \omega_a(x_1,t) .. \omega_a(x_n,t) A_a(x_{n+1},t) .. (-v_{a\alpha}(x_i,t)\partial_\alpha A_a(x_i,t) + \eta J_a(x_i,t)) .. A_a(x_{n+m},t) > . \end{aligned} \tag{7}$$

At each time, the probability law for both the magnetic and velocity fields is modified. We shall suppose that this probability law at time t after a transient period of $O(\tau)$ becomes a statistical field theory described by a Boltzmann weight $\exp(-S_{t,\tau})$

$$d\mathcal{P}(A_a, \psi_a) = \frac{1}{Z} dA d\psi \exp(-S_{t,\tau}(A_a, \psi_a)) \tag{8}$$

where the partition function Z is a normalising factor. The action $S_{t,\tau}$ is a function of $\frac{t}{\tau}$. We shall be interested in the subsequent evolution of this probability law.

The equations (7) can be shown to be invariant under renormalisation transformations. Let us denote by d_A the scaling dimension of A and d_ψ the scaling dimension of ψ obtained

from the field theory (8). These scaling dimensions are well-defined when the field theory (8) is in the basin of attraction of a fixed point of the renormalisation group. In that case, the dimensions d_A and d_ψ are the conformal dimension of the field A and ψ at the fixed point. Let us analyse the solutions of (7). The evolution of solutions of (7) is given by

$$\begin{aligned}\tilde{A}_a(\lambda x, \lambda^T t) &= \lambda^{-d_A} A_{\lambda^{-1}a}(x,t) \\ \tilde{\psi}_a(\lambda x, \lambda^T t) &= \lambda^{-d_\psi} \psi_{\lambda^{-1}a}(x,t)\end{aligned} \tag{9}$$

and for the action

$$e^{-\tilde{S}_{\lambda^T t,\tau}(\tilde{A}_a(\lambda x,\lambda^T t),\tilde{\psi}_a(\lambda x,\lambda^T t))} = \int_{[\frac{a}{\lambda},a]} [dA][d\psi] e^{-S_{t,\frac{\tau}{\lambda^T}}(A_{\lambda^{-1}a}(x,t),\psi_{\lambda^{-1}a}(x,t))} \tag{10}$$

where $\tilde{S}_{\lambda^T t,\tau}$ is the action at time $\lambda^T t$ for the solutions defined by (9). The right-hand side of (10) denotes a renormalisation group transformation where one integrates over all fluctuations of A and ψ in the range $[\frac{a}{\lambda}, a]$. This renormalisation group transformation is carried out in momentum space where modes in the range $[a^{-1}, \lambda a^{-1}]$ are integrated out. We see that the probability law of the stream function remains a statistical field theory given by the renormalised Boltzmann weight (10).

Let us show that (9) and (10) give the evolution of the probability law of the gauge field and the stream function. Let us first write (5) at time t after transforming $a \to \lambda^{-1}a$ and $\tau \to \lambda^{-T}\tau$. This reads

$$\begin{aligned}\partial_t \omega_{\lambda^{-1}a} + e_{\alpha\beta}\partial_\alpha \psi_{\lambda^{-1}a}\partial_\beta \Delta\psi_{\lambda^{-1}a} &= \nu\lambda^{T-2}\Delta\omega_{\lambda^{-1}a} + e_{\alpha\beta}\partial_\alpha J_{\lambda^{-1}a}\partial_\beta A_{\lambda^{-1}a} \\ \partial_t A_{\lambda^{-1}a} + v_{\lambda^{-1}a,\alpha}\partial_\alpha A_{\lambda^{-1}a} &= \eta\lambda^{T-2} J_{\lambda^{-1}a}.\end{aligned} \tag{11}$$

These equations are valid at time t in any correlation functions of the gauge field and the stream function. Using (9), one can substitute $\lambda^{d_A} A_a$ and $\lambda^{d_\psi}\psi_a$ for $A_{\lambda^{-1}a}$ and $\psi_{\lambda^{-1}a}$. In any correlation functions, the functional integral over modes in $[a^{-1}, \lambda a^{-1}]$ can be carried out and involves only the Boltzmann weight. The integration of the Boltzmann weight gives (10). One then obtain the equations

$$\begin{aligned}\partial_{\lambda^T t}\omega_a + \lambda^{2+d_\psi-T} e_{\alpha\beta}\partial_{\lambda x_\alpha}\psi_a \partial_{\lambda x_\beta}\partial^2_{\lambda x}\psi_a &= \lambda^{2+2d_A-d_\psi-T} e_{\alpha\beta}\partial_{\lambda x_\alpha} J_a \partial_{\lambda x_\beta} A_a + \nu\partial^2_{\lambda x}\omega_a \\ \partial_{\lambda^T} t A_a + \lambda^{2+d_\psi-T} v_{a\alpha}\partial_{\lambda x_\alpha} A_a &= \eta J_a.\end{aligned} \tag{12}$$

These equalities are satisfied in any correlation functions at time $\lambda^T t$ and coordinates λx. They coincide with (5) provided the time rescaling exponent and the dimensions of the fields satisfy

$$\begin{aligned}T &= d_\psi + 2 \\ d_A &= d_\psi.\end{aligned} \tag{13}$$

These conditions guarantee that the evolution of the solutions of (7) is given by (9) and (10). The time evolution of solutions is then well-understood when the initial probability law of the gauge field and the stream function (8) is in the basin of attraction of a fixed

point of the renormalisation group. In that case, the probability law of the gauge field and the stream function converges to the probability law defined by the fixed point in the long time regime. Moreover the fixed point is a conformal field theory [6].

Let us analyse the physical consequences of (13). The 2-point correlation functions of the magnetic field and the velocity field is characterised by the scaling dimension (conformal dimension) $d_A = d_\psi$. It is easily seen that in the long time regime, the energy spectra converge to

$$\begin{aligned} E_{\text{mag}}(k) &\sim k^{2d_A+1} \\ E_{\text{kin}}(k) &\sim k^{2d_\psi+1} \end{aligned} \tag{14}$$

in the range $k \ll a^{-1}$. The equality between the scaling dimensions of the gauge field and the stream function implies that these two spectra are proportional. This result characterises the extreme correlation between the velocity field and the stream function. In the following, we shall further analyse the possible fixed points and determine examples of energy spectra.

In the long time regime, solutions describe a turbulent state if the viscosity and the resistivity become negligible. This implies that the dimension of ψ has to be negative in order to ensure that the rescaled viscosity (respectively resistivity) $\nu(\lambda) = \lambda^{d\psi}\nu$ (respectively $\eta(\lambda) = \lambda^{d_\psi}\eta$) deduced from (11) converge to zero. The fixed points are therefore non-unitary conformal field theories. The exponent T being positive, one deduces

$$-2 < d_\psi < 0. \tag{15}$$

The second inequality implies that the dimensions of the electric current J and the vorticity ω are positive. This guarantees that the current J and the vorticity ω decrease sufficiently rapidly at infinity.

Solutions can asymptotically reach a time independent non-unitary fixed point if and only if all the non-linear terms in (11) converge to zero. The non-linear terms can be readily evaluated when the cut-off is rescaled to zero using properties of short distance expansions. Following Polyakov, we shall suppose that the gauge field and the stream function become primary fields at the fixed point with conformal dimensions d_A. Then the short distance expansion of these fields can be summarised using conformal families ($[A]$ is the conformal family of A, i.e. A and all its descendent fields)

$$\begin{aligned} [A][A] &= [A_2] + ... \\ [\psi][\psi] &= [\psi_2] + ... \\ [A][\psi] &= [\chi] + ..., \end{aligned} \tag{16}$$

where dots mean higher order terms. Therefore the non-linear terms read

$$\begin{aligned} e_{\alpha\beta}\partial_\alpha\psi_{\lambda^{-1}a}\partial_\beta\Delta\psi_{\lambda^{-1}a} &\sim (\frac{a}{\lambda})^{d_{\psi_2}-2d_\psi}(L_{-2}\bar{L}_{-1}^2 - \bar{L}_{-2}L_{-1}^2)\psi_{\lambda^{-1}a,2} + ... \\ e_{\alpha\beta}\partial_\alpha J_{\lambda^{-1}a}\partial_\beta A_{\lambda^{-1}a} &\sim (\frac{a}{\lambda})^{d_{A_2}-2d_A}(L_{-2}\bar{L}_{-1}^2 - \bar{L}_{-2}L_{-1}^2)A_{\lambda^{-1}a,2} + ... \\ v_{\alpha,\lambda^{-1}a}\partial_\alpha A_{\lambda^{-1}a} &\sim (\frac{a}{\lambda})^{d_\chi-d_A-d_\psi+2}(L_{-2}\bar{L}_{-1}^2 - \bar{L}_{-2}L_{-1}^2)\chi_{\lambda^{-1}a} + ..., \end{aligned} \tag{17}$$

where $L_n = \frac{1}{2\pi i} \int dz\ z^{n+1} T(z)$ are the Virasoro generators deduced from the energy momentum tensor T of the fixed point. Using the equality between the dimensions of the stream function and the gauge field, these products vanish asymptotically if

$$\begin{aligned} d_{\psi_2} &> 2d_\psi \\ d_{A_2} &> 2d_A \\ d_\chi &> 2d_\psi - 2. \end{aligned} \tag{18}$$

Any non-unitary conformal field theory satisfying these criteria is a time-independent fixed point of the magnetohydrodynamics equations[1].

In order to get more information on the fixed points, we shall be interested in the fluxes of the energy $E = \frac{1}{2} \int d^2x (v_a^2 + B_a^2)$ and the magnetic enstrophies $M_n = \int d^2x A_a^n$. In the Kolmogorov picture[8] of turbulent steady states, these fluxes are constant. In particular, the constraint used by Ferretti and Yang concerns the magnetic enstrophy ($n = 2$). In the decaying case, one can expect these fluxes to vanish in the long time regime as no forcing is imposed. The fluxes associated to the magnetic enstrophies and the energy are [1,2]

$$\begin{aligned} R_{n,a}(q) &= -n \int d^2x\ \theta_q * (v_{a\alpha} \partial_\alpha A_a\ A_a^{n-1}(x)) \\ \epsilon_a(q) &= -\int d^2x \theta_q * (e_{\alpha\beta} \partial_\alpha \psi_a \partial_\beta \Delta \psi_a \psi_a), \end{aligned} \tag{19}$$

where θ_q is an approximate delta function $\theta_q(x) = \int_{k>q} d^2k\ e^{2\pi i k.x}$. The fluxes R_n and ϵ represent the transfer rate of magnetic enstrophies and energy in momentum space. In the asymptotic regime, the q dependence of $R_{n,a}$ disappears. Defining the product of n copies of A with itself, the product of n copies of ψ and the product of n copies of A with ψ [6]

$$\begin{aligned} [A]...[A] &= [A_n] + ... \\ [\psi]...[\psi] &= [\psi_n] + ... \\ [\psi][A]...[A] &= [\chi_{n+1}] + ... \end{aligned} \tag{20}$$

where dots denote higher order terms, the fluxes behave in the long time regime as

$$\begin{aligned} R_{n,a} &= a^{2 + d_{\chi_{n+1}} - (n+1) d_\psi} \int d^2x (L_{-2} \bar{L}_{-1}^2 - \bar{L}_{-2} L_{-1}^2) \chi_{n+1, \lambda^{-1} a} \\ \epsilon_a &= a^{d_{\psi_3} - 3 d_\psi} \int d^2x (L_{-2} \bar{L}_{-1}^2 - \bar{L}_{-2} L_{-1}^2) \psi_{3, \lambda^{-1} a}. \end{aligned} \tag{21}$$

The appearance of a descendent field of χ_{n+1} on the right hand side of (21) stems from the properties of the field χ_{n+1} under parity (similarly for ψ_3). Notice that $r_{n,a}$ is a scalar and χ_{n+1} is a pseudo-scalar as ψ is a pseudo-scalar. The right hand side of (21) is invariant under parity as it is constructed using a combination of Virasoro generators which picks up

[1] The -2 was erroneously missing in Ref.[3][6]

a minus sign under a parity transformation. This combination is the one we had already encountered when calculating the non linear terms of the equations (5). The result (21) is valid provided

$$\begin{aligned} d_{\chi_{n+1}} &= d_{A_{n-1}} + d_\chi \\ d_{\psi_3} &= d_\psi + d_{\psi_2}. \end{aligned} \tag{22}$$

In the decaying case, fluxes given by (21) have vanishing expectation values. Moreover, the fluxes vanish at the fixed points

$$\begin{aligned} R_n &= 0 \\ \epsilon &= 0 \end{aligned} \tag{23}$$

when the dimensions of $(L_{-2}\bar{L}_{-1}^2 - \bar{L}_{-2}L_{-1}^2)\chi_{n+1}$ and $(L_{-2}\bar{L}_{-1}^2 - \bar{L}_{-2}L_{-1}^2)\psi_3$ are different from two. This is expected as there is no forcing on large scales.

In general the constraints (22) are difficult to analyse. We shall construct a particular family of non-trivial fixed points of 2d freely decaying magnetohydrodynamic turbulence. The simplest examples of non-unitary conformal fields can be drawn from minimal models [6]. These models possess a finite number of primary fields. In order to distinguish scalars from pseudo-scalars, it is convenient to introduce a parity factor equal to one for scalars and -1 for pseudoscalars. Let us suppose that the gauge field A becomes equal to the dual of the stream function at the fixed point, i.e. the scalar field $A = A \otimes 1$ differs from the pseudo-scalar $\psi = A \otimes -1$ only by the parity factor. This is compatible with the equality $d_A = d_\psi$. Let us consider now the $(p, p+2)$ non-unitary minimal models ($p > 1$ and odd). One can identify A with the unique field of negative dimension such that $[A][A] = [I] + \ldots$. In particular, one gets

$$d_A = -\frac{3}{2p(p+2)}. \tag{23}$$

This implies that $A_2 = I$, $\psi_2 = I$ and $\chi = \tilde{I}$ where $\tilde{I}$ is the dual of the identity I. The conditions (18) are automatically satisfied. Let us now check the requirements (22). Using the orthogonality between primary fields, one can deduce that $[A][A][A] = [A]$. This entails that $A_n = I$ for n even and $A_n = A$ for n odd. Similarly one obtains $\psi_3 = \psi$, $\chi_{n+1} = \tilde{I}$ for n odd, and $\chi_{n+1} = \psi$ for n even. This implies that (22) is identically satisfied. This proves that $(p, p+2)$ minimal models provide a family of fixed points of 2d magnetohydrodynamics. It is worth noting that in this case the magnetic and the kinetic energy spectra are equal

$$E(k) \sim k^{-\frac{3}{p(p+2)}+1}. \tag{25}$$

This spectrum is always integrable. The $(p, p+2)$ minimal models describe finite energy density solutions of 2d freely decaying magnetohydrodynamic turbulence in the long time regime.

Let us comment on the similarities between the decaying case and steady states of 2d magnetohydrodynamic turbulence. In both situations, conformal field theories are used to describe the probability law of the gauge field and the stream function. For steady states, conformal invariance is an assumption used in order to describe the scale invariant regime in the inertial range. In the freely decaying case, one can control the time evolution of solutions and prove that conformal field theories appear in the long time regime. In both

problems, the vanishing of (17) follows from scale invariance in a regime where the viscosity and the resistivity are negligible. Another similarity is the appearance of non-unitary theories. For steady states, this is a consequence of the constant magnetic enstrophy flux condition. In the free decaying case, the non-unitarity comes from the requirement that the renormalised viscosity $\nu(\lambda) = \lambda^{d_\psi}\nu$ and the renormalised resistivity $\eta(\lambda) = \eta\lambda^{d_\psi}$ become negligible in the long time regime. In both cases, the fact that the viscosity is non-zero plays a fundamental role. This reinforces Polyakov's idea[1] that non-unitary theories should describe turbulence as turbulence is a flux state where dissipation takes place on small scales and not an equilibrium state.

Finally, there are some relevant differences between steady states and the freely decaying case. First of all, solutions of 2d freely decaying magnetohydrodynamic turbulence do not require parity violation whereas it is easier to understand steady states of 2d magnetohydrodynamic turbulence if parity is violated. We have also found that $d_\psi > -2$ at a fixed point. This condition ensures that the vorticity and the electric current decay at infinity. Obviously, this is intimately linked to the absence of boundaries. It is certainly a different situation for steady states where boundary effects play a significant role. The same condition can be written $T > 0$, it is then connected to the breaking of time reversal invariance. Indeed, this constraint implies that solutions at later times $\lambda^T t$ are obtained after averaging over small scale features at time t. Therefore in the process of evolution, small scale details of the gauge field and the stream function are forgotten. This implies that one cannot deduce solutions at time t from solutions at time $\lambda^T t$. In that sense, the behaviour of solutions of magnetohydrodynamics is irreversible. Eventually, the equipartition of magnetic and kinetic energies is not automatically satisfied for steady states. We have seen that the kinetic and the magnetic energy spectra are proportional in the asymptotic regime of the decaying case. Unfortunately, our approach does not allow to determine the proportionality constant relating the kinetic and magnetic energies.

In conclusion, we have studied the long time evolution of 2d freely decaying magnetohydrodynamic turbulence assuming that the initial probability law of the gauge field and the stream function can be described by a statistical field theory in the basin of attraction of a renormalisation group fixed point. We have shown that the time evolution of this probability law is given by renormalisation trajectories. In that case, the long time regime is determined by non-unitary fixed points of the renormalistion group. In the asymptotic regime, the kinetic and the magnetic energy spectra are proportional. We have then constructed a family of fixed points using non-unitary minimal models of conformal field theories. The existence of a large number of fixed points provides an explicit example of non-universality. The convergence towards a particular fixed point is indeed dependent on the initial conditions. It would be extremely interesting to understand the dynamics of 2d magnetohydrodynamic turbulence when the initial conditions are arbitrary. In particular, the influence of the infinite number of fixed points on the trajectories of solutions is a noteworthy problem.

Acknowledgements: I am grateful to W. Eholzer, G. Ferretti, M. Gaberdiel and R. Peschanski for many discussions and suggestions.

References

[1] A. M. Polyakov Nucl. Phys. **B396**, 397 (1993)

[2] G. Ferretti and Z. Yang Europhys. Lett. **22**, 639 (1993)

[3] O. Coceal and S. Thomas " Conformal Models of MHD Turbulence" QMW-PH 95-45 preprint

[4] S. Chandrasekahar Annals of Physics **2**, 615 (1957)

[5] M. R . Rahimi Tabar and S. Rouhani "Turbulent 2D- magnetohydrodynamics and Conformal Field Theory, hep-th/9503005

[6] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov Nucl. Phys. **B241** (1984) 333

[7] G. Ferretti Private Communication

[8] A. N. Kolmogorov: C. R. Aca. Sci. USS **243** (1941) 301.